Modeling for the Equitable and Effective Distribution of Donated Food under Capacity Constraints

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Mathematical models are presented and analyzed to facilitate a food bank’s equitable and effective distribution of donated food among the population at risk for hunger. Typically exceeding the donated supply, demand is proportional to the poverty population within the food bank’s service area. The food bank seeks to ensure a perfectly equitable distribution of food, i.e., each county in the service area should receive a food allocation that is exactly proportional to the county’s demand such that no county is at a disadvantage compared to any other county. This objective often conflicts with the goal of maximizing effectiveness by minimizing the amount of undistributed food. Deterministic network-flow models are developed to minimize the amount of undistributed food while maintaining a user-specified upper bound on the absolute deviation of each county from a perfectly equitable distribution. An extension of this model identifies optimal policies for the allocation of additional receiving capacity to counties in the service area. A numerical study using data from a large North Carolina food bank illustrates the uses of the models. A probabilistic sensitivity analysis reveals the effect on the models’ optimal solutions arising from uncertainty in the receiving capacities of the counties in the service area.

Keywords: Food distribution, equity, fairness, effectiveness, food bank, capacitated network flow problems, resource allocation, humanitarian logistics, food supply chain, food insecurity.

1. Introduction

According to the Food and Agriculture Organization of the United Nations (FAO, 2013), about 870 million people worldwide suffered from food insufficiency and hunger between 2010 and 2012. In 2013, 49.1 million Americans, 15.8 million of whom were children, were exposed to food insecurity, “a household-level economic and social condition of limited or uncertain access to adequate food” (Feeding America, 2015). In this article, we formulate and analyze mathematical models to identify equitable and effective strategies by which a food bank can distribute donated food to the population in need. The food bank is assumed to distribute donated food separately to each county (demand point) in its service area. In this situation, a perfectly equitable distribution of food requires that each county should receive a food allocation in exact
proportion to the county’s demand. We focus on the tactical problem of distributing food to the counties in the food bank’s service area so as to minimize the amount of food that is undistributed, which may be wasted as explained below, while satisfying a user-specified upper bound on the magnitude of each county’s deviation from a perfectly equitable distribution.

In the United States, Feeding America serves as the largest national nonprofit hunger-relief organization with the mission of fighting hunger and food insecurity (Feeding America, 2015). It provides aid to 46.5 million people in the United States from 200 food banks, which are the main food distribution hubs where food donations are collected before being distributed to the beneficiaries. Food banks are autonomous in their operations, except that they are required to report back to Feeding America regarding not only the amount of donated food they distribute in their service areas but also the extent to which they achieve an equitable distribution of that food across the counties in their service areas. For this reason, distributing food donations in an equitable and effective manner is of utmost importance for the continued success of food bank operations.

The Food Bank of Central and Eastern North Carolina (FBCENC), located in Raleigh, NC, is an affiliate of the Feeding America network and exemplifies the type of food bank considered in this article. FBCENC distributes food and other goods through its central warehouse and four branches (located in Wilmington, Durham, Sandhills, and Greenville, NC) to partner agencies such as food pantries and soup kitchens in a 34-county service area, who then distribute the food to the local population in need. The Raleigh branch serves as the headquarters of FBCENC and manages the distribution of food to each county in the service region. Each branch serves a set of counties, and a county may receive food from more than one branch. Food is usually received at the branch location that is most convenient for the donors, but may be transferred from one branch to another prior to distribution to the agencies. FBCENC’s food distribution supply chain is illustrated in Figure 1. The figure shows arcs that are representative of a single branch (Raleigh), but similar flows are possible from all five branches.

FBCENC receives in-kind donations from many sources: 59% of its food is donated by local donors such as grocers, growers, packers, and manufacturers; 21% from state and federal government sources; 11% from Feeding America; 5% from other food banks; and the remaining 4% from food drives (FBCENC, 2013). However, the amount of donated food is persistently much less than the demand in FBCENC’s service area. For example, over the first six months of 2009, donations in FBCENC's service area averaged 3.5 pounds of food per month per person in poverty. According to Feeding America's “Map the Meal Gap” study, the number of additional meals required to meet food needs in North Carolina in 2014 was estimated
to be 175.1 meals per food-insecure person annually (~17.51 pounds of food per food-insecure person monthly) (Feeding America, 2015; FBCENC, 2013).

FBCENC is required to distribute the food that it receives as equitably as possible while minimizing food waste. Food waste may be observed in this network when an agency receives more food than it can store or distribute. If perishable food is not stored properly or is not consumed within its specified shelf-life, then such food becomes unusable and goes to waste. Hence, even though supply is much less than demand, the receiving capacities of the charitable agencies play a significant role in determining the amount of food waste in this network. The receiving capacity of an agency depends on its physical capacity (storage and transportation capabilities) and its financial capacity (budget and workforce) for timely storage and delivery of donated food to its beneficiaries.

**Fig. 1.** Supply chain for the Food Bank of Central and Eastern North Carolina (FBCENC).

In 2013, FBCENC was recognized by Feeding America for its “Fairshare” program, which “uses readily available poverty rates in each county to provide a blueprint of the areas in greatest need” (FBCENC, 2013). FBCENC uses the US Government’s estimate of the population in poverty in each county to guide its food-allocation decisions so that ideally, each county in the service region receives an amount of food approximately equal to the estimated need, relative to that of other counties, for each reporting period. To achieve this goal, FBCENC allocates food to each county according to the estimated size of the poverty population in that county.

In this article, we will also use the estimated size of each county’s poverty population both as an estimate of the county’s population of food-insecure individuals and as a measure of the county’s need for
donated food relative to other counties in the food bank’s service area. In our models, we adopt the definition of an equitable distribution of donated food that has been stipulated by Feeding America and adopted by FBCENC so that for each county, the deviation from perfect equity is the magnitude of the difference between the following:

(a) the size of the county’s poverty population expressed as a fraction of the total poverty population in the food bank’s service area (i.e., the county’s “fair-share” of donated food); and

(b) the amount (weight) of donated food allocated to that county expressed as a fraction of the total amount of food allocated to all counties in the food bank’s service area.

Our use of poverty as a measure of food insecurity is based on widely-available social statistics. For example, it is estimated that 72% of the households served by Feeding America fall at or below the government-defined poverty levels (Feeding America, 2015).

Although FBCENC is among the most successful food banks in achieving its equity goals, historical data suggests that the amount of donated food received per person in poverty can vary substantially across the counties in FBCENC’s service area. In addition, food waste is observed in some locations while others simultaneously face stockouts. The variation in the amount of donated food received per person in poverty at different locations is caused by limited receiving capacity in “bottleneck” locations. FBCENC is often faced with conflicts between its objectives of satisfying the fair-share criterion and not wasting food. For situations in which excess food is available but the counties lacking their fair-share of donated food also lack the capacity to receive additional food, FBCENC will send the excess food to selected counties with sufficient receiving capacity, even though this action may increase the absolute deviations of the selected counties from a perfect-equity allocation. If a county is consistently underserved with respect to that county’s fair-share of donated food, then FBCENC will work to identify the source of the problem and try to fix it by actions such as recruiting new agencies in the county or expanding the receiving capacity of the county’s existing agencies.

In this article we have two primary aims. The first is to obtain policies that help benchmark and improve the performance of food banks by exploring the trade-off between equity and effectiveness of the total food distribution in a capacitated network. Our second aim is to obtain managerial insights into how capacity investments can be made in collaboration with local agencies to improve a food bank’s ability to meet these goals regarding equity and effectiveness. To address these goals, we develop and analyze mathematical models to minimize the amount of undistributed food while maintaining a user-specified upper
bound on the absolute difference between each county’s “fair-share” of donated food and the proportion of total donated food sent to that county. We use data obtained from FBCENC to illustrate our results.

The rest of this article is organized as follows. In the next section we review previous related work. Section 3 introduces the generalized food distribution model, discusses its structure, and derives a closed-form optimal solution of that model. Section 4 focuses on the problem of optimally allocating a fixed amount of additional receiving capacity among the counties in the food bank’s service area. Section 5 presents a case study using data obtained from FBCENC that illustrates our results and examines the sensitivity of optimal solutions to uncertainties or errors in the receiving capacities of the counties in the food bank’s service area. In Section 6, we discuss the conclusions and limitations of our study and we suggest some directions for future research.

2. Previous related work

Celik et al. (2012) divide humanitarian issues into two general categories: disasters and long-term humanitarian development issues. Studies in humanitarian logistics can be classified by the types of decisions (strategic, tactical, and/or operational); objectives and performance measures (cost, equity, sustainability, lead times, effectiveness, etc.); constraints (budget constrained, capacity constrained, time constrained etc.); and reasons for occurrence (natural versus man-made) (Beamon and Balcik, 2008).

We focus on studies of tactical resource allocation in the humanitarian context. Most literature on humanitarian supply chains focuses on disaster-related problems rather than long-term or policy-related problems. While disaster-related issues may have different constraints such as urgency or extremely dynamic donor behavior (Altay, 2012; Balcik et al., 2008; Hwang, 1999), there are similarities with our study, such as the need to coordinate the flow of material from donors to beneficiaries and the objectives to be considered (e.g., equity). Equity, effectiveness, and efficiency are common objectives in this literature and are discussed in a wide variety of contexts. Although the precise meanings of these terms are subjective and depend on the problem and the decision makers (Stone, 1997), for the purposes of this article, we will use equity to mean “the condition of being equal in quantity, amount, value, intensity, etc.” (Simpson and Weiner, 1989). We use the term effectiveness to mean the degree to which something is “capable of being used to a purpose” (Webster’s Third New International Dictionary of the English Language, Unabridged., 1993) and efficiency to mean “getting the most out of a given input” or “achieving an objective for the lowest cost” (Stone, 1997). This paper focuses on the objectives of equity and effectiveness. We will address the objective of equity by limiting the absolute deviation between the proportion of food sent to a
county and that county’s relative need, while effectiveness corresponds to maximizing the total amount of food distributed. Maximizing the total amount of food distributed will also reduce the risk of food being spoiled; increasing the amount of food distributed and reducing food insecurity since higher amounts of healthy, usable food is being distributed in a timely manner. Cost efficiency is not included as a primary objective due to two reasons: (i) assigning transportation costs between branches and counties may result in policies that favor urban locations over rural locations; and (ii) there is limited data available regarding the complex cost structure of this network.

This paper contributes to the literature on humanitarian logistics by addressing the equity-effectiveness trade-off at the tactical level for a nonprofit food chain. Our equity objective, defined above, directly conflicts with the objective of minimizing the amount of unshipped food. Considering the receiving capacities of the counties, we develop policies that minimize the amount of undistributed food for a specified maximum deviation from perfect equity for all counties. We then develop an algorithm to optimally allocate additional capacity to the counties and give the closed-form optimal solution for this problem. We will now explore how the objectives of equity, effectiveness and efficiency are addressed in the literature.

Altay (2012) considers the capabilities of available resources for effective disaster response. Focusing on minimizing response time, he models the case where the available supply exceeds demand in the disaster region. He also considers the case of demand exceeding supply, raising the issues of equity and effectiveness, and formulates a multiobjective optimization problem for minimizing the total deployment time and the total capability deficit, the difference between the resource capability required and that assigned. The total capability deficit addresses the effectiveness objective as in our problem, but equity is not explicitly considered. There are two principal differences between disaster supply chains and the long-term supply chains considered in this paper. First, we do not consider time in our models, since our system requires a continuous flow of goods. Second, in our case the donated supply is much less than the total demand, so satisfying demand is not a useful objective.

Adivar et al. (2010) point out that operations research has been widely used for sudden-onset disaster relief operations but “very little, if any, research focuses on optimizing relief efforts for slow-onset disasters or social welfare operations.” These authors work with an NGO in Turkey to optimize coal distribution by choosing optimal transfer sites and the corresponding shipment amounts using a mixed integer linear program to minimize total cost under various constraints. Celik et al. (2012) study a strategic decision problem based on planning the expansion of a supply chain for banking donated human breast milk in South Africa in order to ensure that every needy infant has equal probability of access to donated breast milk.
Their equity objective is similar to ours, but they focus on a strategic problem (network design) rather than a tactical one (distribution). Malvankar-Mehta and Xie (2012) study the optimal allocation of HIV/AIDS prevention resources considering the complex hierarchical structure of this allocation system. They assume the distribution of these resources is proportional to the size of the infected population in a given region. This is similar to our study, where we estimate demand based on the estimated size of the poverty population in FBCENC’s service region.

The notion of equity is encountered in many different contexts. Sen (1973) states that economic inequality can be described either in an objective sense using statistical measures, or in terms of some normative notion such that a higher degree of inequality corresponds to a lower level of social welfare for a given total income. Although numerous equity measures have been proposed in the literature, there is no single generally accepted equity measure for all problems, making it necessary to select equity measures based on the specific problem (Sen, 1973; Marsh and Schilling, 1994; Leclerc et al., 2012; Balcik et al., 2010). Leclerc et al. (2012) emphasize that the nature of the resource to be allocated, the target group, and the time horizon are the key elements to be considered for the selection of an equity measure.

Marsh and Schilling (1994) discuss twenty equity measures from the literature and review the many areas where equity is used as an objective. Examples of these are geographers’ concerns regarding equitable distribution of water rights in the western United States, political scientists’ discussions of equal representation in Congress, economists’ studies of public welfare distribution and equitable distribution of income. Some studies that consider equity as an objective are Chanta et al. (2011) (determining locations of ambulance dispatch facilities for equitable access by demand zones); Meng and Yang (2002) (optimizing road network expansion for equitable benefits received by people); Mazumdar et al. (1991) (equitable assignment of performance in a multiuser telecommunications network); Wang et al. (2007) (equitable water rights allocation between countries); and Vossen et al. (2003) (equitable allocation of national air space). Marsh and Schilling (1994) focus on the equity objective for facility location problems, defining equity as the case where “each group receives its fair-share of the effect of the facility siting decisions.” This definition is similar to that used in this study, where we define equity to be the case that each county (group) receives its fair-share of the total food donations.

The trade-off between equity and other conflicting objectives has also been studied. Balcik et al. (2014) address the problem of equitable and efficient (low-waste) allocation of food donations through routing and allocation decisions for the donors and agencies of a food bank. Their objective is to maximize the minimum fill rate among all agencies as a measure of equity while implicitly maximizing the amount
of distributed donations. The allocation decisions in their problem are made sequentially at the operational level (the driver decides how much food to allocate to an agency upon arrival), whereas in our study the decisions are made at the tactical level.

Mandell (1991) considers the equity versus effectiveness trade-off in public service delivery systems such as libraries, and develops mathematical models to address this trade-off while using the Gini coefficient as the measure of equity. Leclerc et al. (2012) define effectiveness as “the degree to which a resource allocation causes needs to be met and the extent to which unintended negative impacts of an allocation are avoided.” The second part of this definition supports our usage of the term effectiveness in this study—namely, with reference to minimizing the amount of undistributed food donations. Minimizing the maximum or maximizing the minimum of an equity measure is also common in the equity literature. These approaches seek to improve the overall equity level by improving the condition of the worst-served group (Luss, 1999; Coluccia et al., 2012; Balcik et al., 2014). Although we model equity as a constraint rather than as an objective, we follow this minimax approach to ensure that the maximum deviation from a perfectly equitable food distribution remains below a specified level.

Although there are many studies of for-profit food supply chains (Akkerman et al., 2010; Ahumada and Villalobos, 2009), few studies address nonprofit food distribution supply chains. Cost minimization and food quality are more common objectives in for-profit supply chains, while some typical objectives for nonprofit food supply chains are determining optimal food collection and distribution locations (Davis et al., 2014) and the equitable and effective distribution of food.

Krejci and Beamon (2010) develop supply chain designs to address the problem of maintaining food security worldwide. The authors identify critical characteristics of environmentally sustainable food systems and use these to develop supply chain structures that are environmentally sustainable and provide food security for low, medium, and high customer demand. Schweigman et al. (1990) use operations research tools to address food insecurity in developing countries by determining the optimal amount of land that should be cultivated to prevent food shortage.

Solak et al. (2012) study tactical decision problems arising in nonprofit food distribution networks. They examine a network very similar to ours with food flow from warehouses to distribution sites where they are picked up by agencies for distribution to people in need. They seek to simultaneously optimize three decisions: site selection for food delivery, assignment of agencies to these delivery sites, and routing of delivery vehicles to minimize total transportation costs. None of the mentioned studies considers the equitable and effective allocation of food donations in a nonprofit food chain under capacity constraints.
3. Food distribution model

We present a deterministic linear programming model to achieve optimal allocation of donated food considering objectives of both equity and effectiveness. The distribution of donated food is defined to be \textit{perfectly equitable} if food donations are distributed to the counties such that the fraction of total donated food allocated to a county is exactly equal to the fraction of the total poverty population residing in that county. On the other hand, the distribution is \textit{effective} if the amount of undistributed supply is minimized, minimizing the amount of wasted food by ensuring timely delivery of healthy, usable food to the beneficiaries.

Limited receiving capacity at the partner agencies who distribute the food creates conflict between the objectives of equity and effectiveness. To illustrate the issue, consider a simple supply chain with a single supply node and three demand nodes. The amount of supply at the supply node and the demand and capacities at each demand node are shown in Figure 2. A trivial solution with perfect equity is to send out no food at all, resulting in an equitable but ineffective solution with high waste. Although a zero – allocation solution is not realistic, it is an optimal solution if our sole objective is to achieve perfect equity. On the other hand, in order to maximize effectiveness we should distribute as much supply as the demand nodes can receive, which is equal to their capacities, for a total of 85 units. However, the amount of supply received per person in poverty, given by the capacity to demand ratio for each node, is 0.5 units at Node 1, 1.2 units at Node 2, and 0.35 units at Node 3; increased effectiveness results in a less equitable solution.

![Fig. 2. Example distribution network to illustrate the trade-off between equity and effectiveness.](image)

Although agencies serve as food distribution points for the food-insecure population in FBCENC’s food distribution chain, our models use counties as the smallest distribution points. Therefore, any agency level data is aggregated in terms of the county in which the agency is located. Furthermore, the models
consider the allocation decisions faced by FBCENC in a single time period, such as a month. This is consistent with FBCENC’s practice of distributing available food to the agencies as soon as possible given the perishable nature of the food and the imbalance between demand and supply.

The amount of donated food a county can receive from FBCENC depends on the available storage space, budget, and availability of transportation to the agencies in that county, since the agencies are responsible for picking up food from the food bank. If the donated food is perishable, additional issues such as the availability of refrigerated storage arise. Since FBCENC considers the receiving capacities of the counties to be the major constraint in their supply chain, we consider capacities at the county level. Our models do not assess any cost for interbranch flows since this would penalize counties served from the rural branches over those served from the hub, conflicting with both the equity objective and the actual operation of the food bank. Since we make no distinction between branches in terms of capacity and no-cost interbranch flows are allowed, all five FBCENC branches are aggregated into a single supply node. The assumption of no-cost interbranch flows reflects the actual operations of the food bank since FBCENC collects donations at the branch location that is the most convenient for the donor. If additional food is required for a branch to serve its counties, the hub location (i.e., the Raleigh branch) supplies the extra food to that branch.

Given a set \( J = \{ j: j = 1, \ldots, n \} \) of \( n \) counties, we define the decision variables \( X_j \) to represent the total pounds of food to be shipped to county \( j \), and \( P \) the pounds of undistributed food remaining after all shipments are completed. This model has the following parameters expressed in pounds of food: (i) the total supply \( S \); (ii) the demand \( D_j \) in county \( j \in J \); and (iii) the receiving capacity \( C_j \) of county \( j \in J \). The parameter \( K \), the equity deviation limit, represents the maximum allowed deviation from equity and takes values between zero and one, allowing us to explore the trade-off between equity and effectiveness. The case of \( K = 0 \) is referred to as “perfect equity”. The Food Distribution Model can be written as follows:

\[
\begin{align*}
\text{minimize} & \quad P \\
\text{subject to} & \quad \text{...} 
\end{align*}
\]
\[
\left| \frac{X_j}{\sum_{l=1}^{n} X_l} - \frac{D_j}{\sum_{l=1}^{n} D_l} \right| \leq K \quad j \in J
\]  
(2)

\[
S - \sum_{i=1}^{n} X_i - P = 0
\]  
(3)

\[
X_j \leq C_j \quad j \in J
\]  
(4)

\[
X_j, P \geq 0 \quad j \in J.
\]  
(5)

Constraint (2) ensures that the proportion of all distributed food received by county \(j\) deviates from the proportion of total demand represented by county \(j\) by no more than the deviation limit \(K\). Constraint (3) enforces conservation of flow at the branch level and constrains the total distribution by the amount of available supply. Constraint set (4) is the set of receiving capacity constraints for each county \(j\). Constraints (5) are nonnegativity constraints. The objective function (1) minimizes the total undistributed supply over all branches. Constraints (2) are equivalent to:

\[
-K \leq \frac{X_j}{\sum_{l=1}^{n} X_l} - \frac{D_j}{\sum_{l=1}^{n} D_l} \leq K \quad j \in J,
\]  
(2.a)

which can be linearized as follows:

\[
X_j \sum_{l=1}^{n} D_l - D_j \sum_{l=1}^{n} X_l + K \sum_{l=1}^{n} X_l \sum_{l=1}^{n} D_l \geq 0 \quad j \in J,
\]  
(6)

\[
X_j \sum_{l=1}^{n} D_l - D_j \sum_{l=1}^{n} X_l - K \sum_{l=1}^{n} X_l \sum_{l=1}^{n} D_l \leq 0 \quad j \in J.
\]  
(7)

Since (2) is mathematically equivalent to the following:

\[
\max_{j \in J} \left| \frac{X_j}{\sum_{l=1}^{n} X_l} - \frac{D_j}{\sum_{l=1}^{n} D_l} \right| \leq K,
\]  
(2.b)

equity among counties is addressed by limiting the maximum absolute deviation from a perfectly equitable solution through the equity deviation limit \(K\). Minimizing the maximum absolute deviation from perfect equity is appropriate for our problem since this improves the condition of the county that receives the worst service in terms of equity, improving overall equity in the entire service region. By modeling equity as a
constraint, we can control the level of inequity permitted in the system while preserving the convexity of the feasible region.

It is evident from the formulation that the structure of the optimal solution is defined by the interactions of the total available supply $S$, the receiving capacities $C_j$ and demands $D_j$ of the counties. To this end we define three important ratios. The first of these is $DF_j = \frac{D_j}{\sum_{l=1}^{n} D_l}$, the fraction of total demand incurred in that county. The second is the capacity to demand (CD) ratio of each county $CD_j = \frac{C_j}{D_j}$. This ratio relates capacity to demand, but does not consider equity in any way. In order to incorporate equity, we define the modified capacity to demand (MCD) ratio $MCD_j = \frac{C_j}{D_j - K \sum_{l=1}^{n} D_l}$. Based on these, the set

$$J_0 = \left\{ j \in J \mid \frac{D_j}{\sum_{l=1}^{n} D_l} = DF_j > K \right\}$$

defines the set of counties $j$ that must receive a nonzero allocation of food ($X_j > 0$) to satisfy the equity constraints (2). In addition, if

$$R = \min_{j \in J_0} \left\{ \frac{C_j}{D_j - K \sum_{l=1}^{n} D_l} \right\} = \min_{j \in J_0} \{ MCD_j \},$$

we can define the set

$$B = \left\{ j \in J_0 \mid \frac{C_j}{D_j - K \sum_{l=1}^{n} D_l} = R \right\}$$

as the set of bottleneck counties. Recall that the equity constraints require that the fraction of available food sent to county $j$, $\frac{X_j}{\sum_{l=1}^{n} X_l}$, deviates from the fraction of total demand required by that county, $\frac{D_j}{\sum_{l=1}^{n} D_l}$, by no more than the specified equity limit $K$. Proposition 1 shows that the counties in the set $B$ constrain the amount of food shipped to other counties with higher MCD$_j$ ratios.

Given these preliminaries, Proposition 1 defines an optimal solution of the Food Distribution Model under all three possible cases that can arise in a problem instance. For this proposition, we assume that $\sum_{l=1}^{n} C_l \geq S$. This assumption is not limiting since if $S > \sum_{l=1}^{n} C_l$, at least $S - \sum_{l=1}^{n} C_l$ units of supply will go to waste. In that case, we can set $S = \sum_{l=1}^{n} C_l$ and the proposition will hold.

**Proposition 1.** An optimal solution to a given instance of the Food Distribution Model is as follows:
Case 1: If $\sum_{l=1}^{n} C_l \geq S$ and $J_0 = \emptyset$, i.e., the instance is partially equity constrained, then the optimal objective function value is:

$$P^* = 0,$$

(11)

and the individual distributions to the counties $(X_j^*)$ have multiple optimal solutions.

Case 2: If $J_0 \neq \emptyset$ and $\sum_{l=1}^{n} C_l \geq S \geq R \sum_{l=1}^{n} D_l$, i.e., the instance is capacity and equity constrained, then the optimal objective function value is:

$$P^* = S - R \sum_{l=1}^{n} D_l.$$

(12)

In an optimal solution, the bottleneck counties receive an amount of food equal to their capacity,

$$X_j^* = C_j \text{ for all } j \in B.$$

(13)

Food shipments to all remaining counties will have multiple optimal solutions.

Case 3: If $J_0 \neq \emptyset$, $\sum_{l=1}^{n} C_l \geq S$, and $R \sum_{l=1}^{n} D_l > S$, i.e., the instance is supply and equity constrained, then the optimal objective function value is:

$$P^* = 0.$$

(14)

The individual distributions to the counties $(X_j^*)$ will have multiple optimal solutions.

Proof. Given in Appendix A.

In Case 1, the equity constraints do not prevent the entire supply from being shipped since every county has $K \geq \frac{D_j}{\sum_{l=1}^{n} D_l}$. In Cases 2 and 3 the set $J_0$ is nonempty, implying that the counties $j \in J_0$ will limit the total distribution. In Case 3 there is sufficient capacity for all supply to be distributed, while in Case 2 the equity constraints prevent all supply from being distributed.

The key insight from Proposition 1 is the critical role of the MCD ratios in determining the structure of the solution. The MCD ratios combine information on capacity, equity and demand into a quantitative measure of the equity-effectiveness trade-off. Very low MCD ratios may be due to low capacities or a low equity deviation limit. In either case, the optimal food distribution is also low, resulting in high waste. Proposition 1 also highlights the fact that all three cases result in multiple optimal solutions for the individual flows $X_j$ to the nonbottleneck counties. We now introduce a Multiple Optima Generation (MOG) algorithm for generating alternative optimal solutions in Figure 3. This algorithm uses insights from Proposition
1 and its proof to examine alternatives for allocating distributed food among the counties in the service region while shipping the maximum possible amount and maintaining the specified level of equity. The correctness of the algorithm follows directly from the proof of Proposition 1.

After necessary definitions and initialization, MOG Algorithm determines which case of Proposition 1 applies to the problem instance at hand. If Case 2 applies, all bottleneck counties will receive food equal to their capacity. Once the demand of the bottleneck counties has been met in this fashion, a feasible solution to the Food Distribution Model satisfying $\sum_{j=1}^{n} X_j^* = R \sum_{i=1}^{n} D_i$ as shown in Case 2 of Proposition 1 is given by $X_j^* = R D_j$. If $RD_j < C_j$, we set $X_j^* = R D_j$ and add county $j$ to the set $J_E$ of counties with excess capacity. After this is done for all $j \in J$, we sort the indices $j \in J_E$ in random order. Then, for all $l \in J \setminus J_E$, we set $X_l^* = C_l$ and calculate the amount of remaining food $E_l = RD_l - C_l$ that must be reallocated from county $l$ in order to achieve the optimal total distribution. We then distribute these extra $E_l$ pounds of food among the counties $j \in J_E$ without violating any capacity constraints. This is done by allocating $E_l$ pounds of food to any county $j \in J_E$ with excess capacity until its capacity is saturated and moving on to the next county with excess capacity, until all $E_l$ pounds of food have been allocated. This allocation scheme results in an optimal solution as discussed in the proof of Proposition 1.

The second part of the algorithm considers Cases 1 and 3 of Proposition 1, the supply constrained instances. In these cases a solution to the Food Distribution Model satisfying $\sum_{j=1}^{n} X_j^* = S$ is given by $X_j^* = \frac{SD_j}{\sum_{i=1}^{n} D_i}$ and we use this as a basis to assign food to the counties. Different orderings of the set $J_E$ yield different optimal solutions to the Food Distribution Model. The worst-case time complexity of the MOG Algorithm is $O(n^2)$.

In order to illustrate the operation of the MOG Algorithm, consider the network introduced in Figure 2 with $K = 0.01$. This instance is capacity and equity constrained since $R \sum_{i=1}^{n} D_i = 58.718 < S = 100$, and hence has multiple optimal solutions by Proposition 1, with county 3 being the bottleneck county. Table 1 shows two alternative optimal solutions to this instance that suggest different shipments to counties 1 and 2 and maintain the same total distribution.
Multiple Optima Generation (MOG) Algorithm

\[ \Delta = \sum_{l=1}^{n} D_l, J_0 \leftarrow \{ j \in J \mid \frac{C_j}{D_j} > K \}, R = \min_{j \in J_0} \left\{ \frac{C_j}{D_j} \right\}, B = \{ j \in J_0 \mid \frac{C_j}{D_j} = R \} \]

\[ X_j \leftarrow 0, E_l \leftarrow 0, \forall j \in J \]

\[ j_E \leftarrow \phi, B_E \leftarrow \phi \]

if \( R \Delta \leq S \) \& \& \( |J_0| > 0 \) then

for \( j = 1 \) to \( n \) do

if \( j \in B \) then

\[ X_j \leftarrow C_j \]

else if \( RD_j \leq C_j \) then

\[ X_j \leftarrow RD_j, J_E \leftarrow J_E \cup \{ j \} \]

end if

end if

end for

Sort \( j \in J_E \) in a random order

for \( l = 1 \) to \( n \) do

if \( l \notin J_E \) then

\[ X_i \leftarrow C_i, E_i \leftarrow RD_i - C_i, B_E \leftarrow B_E \cup \{ l \} \]

for all \( j \in J_E \) do

if \( X_j + E_i \leq C_j \) then

\[ X_j \leftarrow X_j + E_i, E_i \leftarrow 0, \text{ Continue to next } l \]

else

\[ E_i \leftarrow E_i - (C_j - X_j), X_j \leftarrow C_j, B_E \leftarrow B_E \cup \{ j \} \]

end if

end for

end if

end for

else

for \( j = 1 \) to \( n \) do

if \( \frac{SD_j}{\Delta} \leq C_j \) then

\[ X_j \leftarrow \frac{SD_j}{\Delta}, J_E \leftarrow J_E \cup \{ j \} \]

end if

end for

Sort \( j \in J_E \) in a random order

for \( l = 1 \) to \( n \) do

if \( l \notin J_E \) then

\[ X_i \leftarrow C_i, E_i \leftarrow \frac{SD_i}{\Delta} - C_i, B_E \leftarrow B_E \cup \{ l \} \]

for all \( j \in J_E \) do

if \( X_j + E_i \leq C_j \) then

\[ X_j \leftarrow X_j + E_i, E_i \leftarrow 0, \text{ Continue to next } l \]

else

\[ E_i \leftarrow E_i - (C_j - X_j), X_j \leftarrow C_j, B_E \leftarrow B_E \cup \{ j \} \]

end if

end for

end if

end for

end if

Fig. 3. Multiple Optima Generation (MOG) Algorithm.

Table 1. Alternative optimal solutions to the simple network in Figure 2.
Corollary 1 applies Proposition 1 to the perfect equity case of $K = 0$.

**Corollary 1.** Let $K = 0$ and assume that $\sum_{l=1}^{n} C_l \geq S$. In the optimal solution to the Food Distribution Model, if $S \geq R \sum_{l=1}^{n} D_l$, i.e., the solution is capacity constrained, then $X_j^* = C_j$ for $j \in B$ and $X_j^* < C_j$ for $j \notin B$. On the other hand, if $R \sum_{l=1}^{n} D_l > S$, i.e., the solution is supply constrained, then $X_j^* < C_j$ for all $j$.

The optimal solution is given by:

$$X_j^* = \min \left\{ S \frac{D_j}{\sum_{l=1}^{n} D_l}, RD_j \right\} \quad j \in J \quad (15)$$

$$P^* = \max \left\{ 0, S - R \sum_{l=1}^{n} D_l \right\} \quad (16)$$

For the supply constrained perfect equity case, supply is distributed among the counties according to their DF ratios, with each county receiving less food than its capacity. If the instance is capacity constrained, the bottleneck counties $j \in B$ receive food equal to their capacities, while the remaining counties $j \in J \setminus B$ will have excess capacity. Therefore the optimal effectiveness under perfect equity is determined by the minimum CD ratio among all counties. Under perfect equity, Constraint (2) becomes $X_j = \frac{D_j}{\sum_{l=1}^{n} D_l} \sum_{l=1}^{n} X_l$, yielding a unique optimal solution. If the instance is capacity constrained, then $X_j^* = RD_j \leq C_j$ for all $j \in J$ by definition and $\sum_{l=1}^{n} X_l^* = R \sum_{l=1}^{n} D_l$. On the other hand, if the instance is supply constrained, then $X_j^* = S \frac{D_j}{\sum_{l=1}^{n} D_l} < RD_j \leq C_j$ for all $j \in J$ by assumption and $\sum_{l=1}^{n} X_l^* = S$.

Since demand is treated as exogenous to the model, Proposition 1 and Corollary 1 imply that for a capacity constrained instance, the amount of food distributed cannot be increased while maintaining equity unless the capacities of the bottleneck counties are increased. We now examine how to increase the capacities of the bottleneck counties to increase distribution effectiveness while preserving equity.
4. Capacity allocation problem

A problem of interest to FBCENC is how to allocate a given amount of additional capacity among the counties such that equity is maintained and effectiveness is increased. This additional capacity can be in the form of storage capacity like extra freezer space, money to increase an agency’s purchasing budget, opening a new agency in a county, or increasing distribution capacity by means such as mobile pantries or additional delivery vehicles. FBCENC often solicits grant or donor funding to purchase such additional capacity to enable agencies to distribute more food in their areas. Our results in this section are helpful to the food banks in developing proposals for funding to external sponsors by quantifying the benefit of additional capacity in terms of the additional food that can be distributed.

We now develop a linear programming model to determine the optimal allocation of additional capacity to the counties that will maximize the increase in effectiveness while maintaining the desired level of equity. We do not consider any costs associated with the allocation of additional capacity to counties, as our focus is to identify priorities for the allocation of additional capacity. We only consider counties in set $J_0$ since only these counties have the potential to make the problem capacity constrained as discussed in Section 3.

Without loss of generality, we assume the counties in set $J_0$ are indexed in increasing order of their unique MCD ratios, i.e., for $j, g \in J_0$ and $j < g$, we have $\frac{C_j}{D_j - K \sum_{l=1}^{n} D_l} < \frac{C_g}{D_g - K \sum_{l=1}^{n} D_l}$. The assumption of unique MCD ratios is not restrictive since both capacity and demand are expressed in tons of food, and are hence large, continuous quantities. In the Capacity Allocation Model, the only new parameter we introduce is $Y$, the amount of extra capacity available to the food bank, in pounds of food. The decision variable $\rho_j$ represents the fraction of the additional capacity $Y$ allocated to county $j$, and $Q$ the minimum MCD ratio among the counties after the additional capacity has been allocated. We formulate the Capacity Allocation Model as follows:

$$\text{maximize } Q$$

subject to

$$Q \leq \frac{C_j + \rho_j Y}{D_j - K \sum_{l=1}^{n} D_l} \quad j \in J_0$$
\[
\sum_{l \in J_0} \rho_l \leq 1 \quad (19)
\]

\[Q, \rho_j \geq 0 \quad j \in J_0. \quad (20)\]

The objective of this model is to maximize the minimum MCD ratio, i.e., the value of the MCD ratio of all the bottleneck counties, as implemented in the objective function (17) and constraints (18). Constraint (19) ensures that the total capacity distributed cannot exceed the available additional capacity \(Y\). Constraints (20) are nonnegativity constraints.

We propose an efficient iterative algorithm to solve the Capacity Allocation Model exactly, and show that this can be used to obtain a closed-form solution. The algorithm is based on the insight from Proposition 1: the total amount shipped cannot be increased while preserving equity unless the MCD ratio of the bottleneck counties is increased. In other words, since the problem is to maximize the minimum MCD ratio, the algorithm tries to maximize the number of bottleneck counties with the same MCD ratio. Thus at each iteration the algorithm identifies the MCD ratio of the current bottleneck counties, and the county with the next lowest MCD ratio, which we shall call the target MCD ratio. It then tries to allocate additional capacity to all current bottleneck counties to raise their MCD ratios to the target MCD ratio. If this is successful, the county associated with the target MCD ratio is added to the set of bottleneck counties and a new iteration begins. If there is insufficient additional capacity available to accomplish this, any as yet unallocated additional capacity is allocated among the current bottleneck counties such that they all have equal MCD ratios, and the algorithm terminates. We now give a more detailed description of the algorithm, which is formally stated in Figure 4.

For a given iteration \(z\), let \(R_z\) denote the minimum MCD ratio at that iteration, \(T_z\) the second-lowest MCD ratio, and \(B_z = \{j \in J | \text{MCD}_j = R_z\}\) the current set of bottleneck counties. The variable \(Y_z\) denotes the amount of capacity in pounds remaining to be distributed in the beginning of iteration \(z\).

The algorithm is initialized with \(z = 1\), \(B_1 = \{1\}\) and \(Y_1 = Y\). At each iteration \(z\), we first check whether the maximum number of iterations \(|J_0|\) has been reached. If not, we determine \(R_z, B_z, T_z\) and the associated county \(u\) with \(\text{MCD}_u = T_z\). We also calculate \(\varphi_z\), the amount of additional capacity that must be allocated to increase the capacities of the counties \(j \in B_z\) such that their MCD ratios become equal to \(T_z\). Note that counties in \(B_z\) will not receive equal allocations of additional capacity, since they have different capacities and demands. If \(Y_z \geq \varphi_z\), we distribute the \(\varphi_z\) units of capacity among the counties in set \(B_z\)
such that their revised MCD ratios are equal to $T_z$. In Figure 4, for each iteration $z$ of the algorithm and each county $j \in B_z$, $\rho_j$ represents the fraction of the additional capacity $Y$ allocated to county $j$ prior to iteration $z$ so that $C_j + \rho_j Y$ is the allocated capacity of county $j$ just before iteration $z$ is performed. If we let $W_l^z$ denote the additional capacity to be allocated to county $l$ at iteration $z$, this means that the revised MCD ratios for the counties in $B_z$ must satisfy

<table>
<thead>
<tr>
<th>Capacity Allocation Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta = \sum_{l=1}^{n} D_l$</td>
</tr>
<tr>
<td>$\rho_j \leftarrow 0, \forall j \in J_0$</td>
</tr>
<tr>
<td>$z \leftarrow 1, B_z \leftarrow {1}, Y_z \leftarrow Y$</td>
</tr>
<tr>
<td><strong>while</strong> $z \leq</td>
</tr>
<tr>
<td><strong>if</strong> $z \neq</td>
</tr>
<tr>
<td>$R_z \leftarrow \min_{j \in J_0} \left{ \frac{C_j + \rho_j Y}{\gamma} \right}, B_z \leftarrow \left{ j \in J_0 \mid \frac{C_j + \rho_j Y}{\Delta} = R_z \right}$</td>
</tr>
<tr>
<td>$T_z \leftarrow \min_{j \in J_0 \setminus B_z} \left{ \frac{C_j}{\gamma} \right}, B_{z+1} \leftarrow \left{ j \in J_0 \mid \frac{C_j}{\gamma} = T_z \right}$</td>
</tr>
<tr>
<td>$\varphi_z \leftarrow \sum_{l \in B_z} \left( T_z D_l - C_l - \rho_l Y - KT_z \Delta \right)$</td>
</tr>
<tr>
<td><strong>if</strong> $Y_z \geq \varphi_z$ <strong>then</strong></td>
</tr>
<tr>
<td>$\rho_j \leftarrow \frac{T_z D_l - C_l - KT_z \Delta}{Y}, \forall j \in B_z$</td>
</tr>
<tr>
<td>$Y_{z+1} \leftarrow Y_z - \varphi_z, z \leftarrow z + 1$</td>
</tr>
<tr>
<td><strong>else</strong></td>
</tr>
<tr>
<td>$R_z \leftarrow \frac{Y_z + \sum_{l=1}^{z} (C_l + \rho_l Y)}{\sum_{l=1}^{z} (D_l - \Delta)}, B_{z+1} \leftarrow \left{ j \in J_0 \mid (D_l - \Delta) \right}$</td>
</tr>
<tr>
<td>$\rho_j^* \leftarrow \frac{R_z D_l - C_l - KT_z \Delta}{Y}, \forall j \in J_0$</td>
</tr>
<tr>
<td>$Q^* \leftarrow R_z$</td>
</tr>
<tr>
<td>$\eta \leftarrow z$</td>
</tr>
<tr>
<td>Quit</td>
</tr>
<tr>
<td><strong>end if</strong></td>
</tr>
<tr>
<td><strong>end if</strong></td>
</tr>
</tbody>
</table>

**Fig. 4.** Capacity Allocation Algorithm.
\[
\frac{C_l + \rho_l Y + W_l^z}{D_l - K\Delta} = T_z \quad l \in B_z,
\]  

(21)

where \( \Delta = \sum_{i=1}^{n} D_l \), from which it follows immediately that we must have

\[
W_l^z = T_z D_l - C_l - \rho_l Y - KT_z \Delta \quad l \in B_z;
\]

(22)

and since by definition

\[
\varphi_z = \sum_{l \in B_z} W_l^z,
\]

(23)

the assignment

\[
\varphi_z \leftarrow \sum_{l \in B_z} (T_z D_l - C_l - \rho_l Y - KT_z \Delta)
\]

(24)

in Figure 4 must hold. Notice that \( B_{z+1} = B_z \cup \{u\} \), increasing the number of bottleneck counties by one.

If \( Y_z < \varphi_z \), then there is not enough unallocated capacity remaining to move to iteration \( z + 1 \). In this case, we distribute the remaining capacity on hand among the counties in set \( B_z \) such that their MCD ratios (the current \( R_z \)) increase but remain equal to each other. We set \( Q = R_z \) and terminate the algorithm.

If we reach iteration \( |J_0| \), we distribute the remaining capacity on hand among all counties such that the MCD ratios remain equal to each other, set \( Q = R_z \) and terminate. This case occurs only if \( Y \) is sufficiently large to make every county’s MCD ratio equal to the maximum MCD ratio. The worst-case time complexity of the Capacity Allocation Algorithm is \( O(n^2) \).

In order to prove the optimality of the Capacity Allocation Algorithm, we first show that in the optimal solution to the Capacity Allocation Model, all additional capacity \( Y \) will be distributed.

**Proposition 2.** In the optimal solution to the Capacity Allocation Algorithm, Constraint (19) will be satisfied at strict equality.

**Proof.** Given in Appendix B.

Proposition 3 outlines the structure of the optimal solution for the Capacity Allocation Model.

**Proposition 3.** In the optimal solution to the Capacity Allocation Model, there will be \( \eta \) bottleneck counties where
\[
\eta = \arg\max_{1 \leq \xi \leq |J_0|} \left\{ \sum_{i=1}^{\xi-1} \left( \frac{C_\xi (D_i - K\Delta)}{D_\xi - K\Delta} - C_i \right) \leq Y \right\}
\tag{25}
\]

and the summation in (25) is taken to be zero when \( \xi = 1 \). The optimal solution is given by:

\[
Q^* = \frac{Y + \sum_{g=1}^{\eta} C_g}{\sum_{i=1}^{\eta} (D_i - K\Delta)}
\tag{26}
\]

\[
\rho_j^* = \begin{cases} 
Q^* (D_j - K\Delta) - C_j & \text{for } 1 \leq j \leq \eta, \\
0 & \text{for } \eta + 1 \leq j \leq |J_0|.
\end{cases}
\tag{27}
\]

**Proof.** Given in Appendix C.

Corollary 2 follows from Proposition 3 for the perfect equity case with \( K = 0 \).

**Corollary 2.** Let \( K = 0 \). Then, in the optimal solution to the Capacity Allocation Model, there will be \( \eta \) bottleneck counties where

\[
\eta = \arg\max_{1 \leq \xi \leq n} \left\{ \sum_{i=1}^{\xi-1} \left( \frac{C_\xi D_i}{D_\xi} - C_i \right) \leq Y \right\}
\tag{28}
\]

The optimal solution is given by:

\[
Q^* = \frac{Y + \sum_{g=1}^{\eta} C_g}{\sum_{i=1}^{\eta} D_i},
\tag{29}
\]

\[
\rho_j^* = \begin{cases} 
Q^* D_j - C_j & \text{for } 1 \leq j \leq \eta, \\
0 & \text{for } \eta + 1 \leq j \leq n.
\end{cases}
\tag{30}
\]

Proposition 3 and Corollary 2 provide closed form solutions to the Capacity Allocation Model. The fraction of the additional capacity a county receives is an increasing linear function of its demand and a decreasing linear function of its capacity, so that more densely populated areas receive more capacity, as expected. The additional capacity \( Y \) on hand is a parameter in this formulation; if the value of \( K \) is also known, then the increase in the distribution effectiveness (total food distributed) can be calculated directly using the results from Propositions 1 and 3. These ideas are illustrated in the next section.
5. Computational results and discussion

5.1. A case study

In this section we use data obtained from FBCENC to illustrate the findings from the previous sections. The results are useful for understanding the structure of FBCENC’s nonprofit supply chain, highlighting problem areas in this system and suggesting policies to be applied under different scenarios.

FBCENC classifies donated goods into four categories: dry goods, produce, refrigerated food and frozen food. This paper focuses on a single food category (dry goods) which constitutes the largest category in terms of the amount of food donations received by FBCENC. However, our models could be implemented for other food categories as well. The models are solved using IBM ILOG Optimization Programming Language (IBM, 2013).

The actual donations made to FBCENC’s five branches during January 2009 constitute the supply data which, since our models assume a single supply node, is aggregated over all five branches. The donations data are given in pounds of food. The total amount of donated dry goods during January 2009 was 1,277,363 pounds and corresponds to approximately 45% of all donations during this period.

In this problem, it is not possible to predict demand with certainty although it is reasonable to assume it is proportional to the poverty populations in the counties. Poverty itself is not a fixed, universally-understood concept: families and individuals may move above and below defined poverty thresholds frequently, and deprivation may be acute though family income may exceed the poverty threshold. The poverty populations of the counties for the year 2009 are obtained from Census data (United States Census Bureau, 2009).

FBCENC does not have exact data on the receiving capacities of the counties in its service region. However, it is reasonable to use the historical amount of food sent to the counties to estimate the counties’ capacities based on how much each county was able to receive in the past. The 90th percentile of the empirical distribution of the amount of food shipped to that county each month during fiscal year 2009 is used as an estimate of each county’s capacity. The 90th percentile statistic was chosen because a value belonging to the upper quartile of the data was desired so it represents what the county is able to receive. However, it should not be as extreme as the maximum since using the maximum could result in overestimating a county’s capacity and waste of allocated food.

We first show the results for the perfect equity case and then illustrate how the optimal solution changes as we increase the equity deviation limit $K$. Setting $K = 0$ in the Food Distribution Model requires
that each county receives their exact relative need from the total food distribution according to the demand residing in that county. In Figure 5, the horizontal axis shows the 34 counties in the FBCENC service region in increasing order of their capacity to demand (CD) ratios. The dark grey line shows the ratios of the shipments $X_j$ received by the counties to their capacities $C_j$ on the primary vertical axis. The bars represent the CD ratios of the counties with their values on the secondary vertical axis. The shipment to capacity ratio for the bottleneck county, Wilson, is equal to one. All the other counties receive less food than their capacities, and hence are penalized by the requirement of perfectly equitable distribution across counties with different capacities.

![Figure 5](image)

**Fig. 5.** The results from the Food Distribution Model for $K = 0$.

To explore the relationship between equity and effectiveness, we use the Food Distribution Model to construct the efficient frontier between the undistributed supply and the permitted deviation from a completely equitable solution. The total undistributed supply decreases as $K$ is increased. Figure 6 illustrates this relationship where $K$ is varied between zero and 0.01 with increments of 0.001. We observe that a value of $K = 0.0047$ results in the distribution of all the supply. For $K > 0.0047$ the solution becomes supply constrained as explained in Section 3. Thus, permitting a slight deviation from equity of less than 1% results in considerable decrease in waste.
Fig. 6. $P^*$ Solutions from the Food Distribution Model for increasing $K$.

We illustrate the results from the Capacity Allocation Model for $K = 0$ in Figures 7a and 7b. The horizontal axes of Figures 7a and 7b list the counties in increasing order of their CD ratios. Figure 7a shows the cumulative additional capacity required for the corresponding county to be added to the set of bottleneck counties. For example, if there are 500,000 pounds of additional capacity to be allocated, we can use this additional capacity to increase the receiving capacities of the counties resulting in an increase of the minimum CD ratio from 0.075, which is Wilson county’s CD ratio, to 0.150. From Figure 7a, we see that 500,000 pounds of additional capacity results in 20 counties (Wilson, Granville, Wayne, Sampson, Onslow, Halifax, Orange, Lenoir, Wake, Craven, Warren, Carteret, Pitt, Johnston, Durham, New Hanover, Person, Greene, Nash and Columbus) becoming bottlenecks that can receive food up to their capacity. Franklin County cannot be included because 559,986 pounds of total additional capacity would be required to include it, as can be seen from the graph. Using (26) we can find the resulting minimum CD ratio such that $Q^* = \frac{500,000 + \sum_{g=1}^{20} c_g}{\sum_{t=1}^{20} D_t} = 0.150$. In Figure 7b, the dark grey bars show the optimal amounts of food that can be distributed when additional capacity is allocated as in Figure 7a and the supply constraint is not considered. As the number of bottleneck counties and their capacities increase, distribution effectiveness increases while equitable distribution is maintained. The light grey bars represent the optimal amount of food that can be distributed when additional capacity is allocated as shown in Figure 7a under the supply constraint. As expected, once capacities are increased beyond some point the amount of food distributed starts to be constrained by supply rather than the capacities.
This analysis allows FBCENC to see how they could benefit from increasing their capacities and supply levels and compare different scenarios. For example, consider the case where there are 2,500,000 additional pounds of capacity available for distribution to the counties. Figure 7a shows that this amount of extra capacity is sufficient to achieve equal CD ratios for all counties, resulting in 34 bottleneck counties. Therefore, all counties receive a positive amount of additional capacity. Due to the definition of the optimal fraction of additional capacity $\rho_j^*$ received by county $j$ in (30), we would expect densely populated counties with relatively low capacities to obtain the largest proportion of the extra capacity. The data supports this structure such that two counties in the region with highest demands (Wake and Durham) receive the highest
proportion of total capacity: Wake county receives 18 percent and Durham county 9 percent of the additional capacity.

5.2. Generalizability of the solutions

In Section 3, we showed that the optimal value of the Food Distribution Model is determined by the minimum MCD ratio of the counties. Therefore, any change in the counties’ capacities may directly affect the optimal distribution policy. In Section 5.1, we used data obtained from FBCENC to calculate nominal estimates of county capacities. However, in reality these capacities are neither deterministic nor constant over time; instead, they vary randomly as individual agencies in the counties add or lose capacity. In this section we explore the sensitivity of our solutions to uncertainty in the capacity values. Some of the questions that we would like to answer are: How well do the nominal solutions obtained in Section 5.1 behave under uncertain capacity? Which counties are at greatest risk of becoming bottleneck counties? How does the amount of undistributed supply change when the county capacities follow different probability distributions? We examine these issues by simulating the behavior of the solutions obtained by our model under different random errors in the nominal values of the county capacities.

5.2.1. Design of the probabilistic sensitivity analysis

In order to generate random capacities for each county, we use four different probability distributions which we will refer to as Unif1, Unif2, Beta1 and Beta2. We assume that the counties’ capacities are independent of each other. Let \( C_j^{\text{nom}} \) represent the nominal capacity value for county \( j \) as used in Section 5.1. We set \( C_j \sim \text{Uniform} \left( (1 - \theta)C_j^{\text{nom}}, (1 + \theta)C_j^{\text{nom}} \right) \) where \( \theta = 0.1 \) for the Unif1 distribution and \( \theta = 0.2 \) for the Unif2 distribution. We also use two Beta distributions to capture the changes in the optimal solutions for a skewed error distribution. We set the Beta distribution ranges such that \( C_j \in (a = 0.5 C_j^{\text{nom}}, b = 1.15 C_j^{\text{nom}}) \). These parameters are chosen to obtain a negative skewness based on the 2009 fiscal year FBCENC data; on average the minimum amount of food shipped to a county that year was about half of the 90th percentile, and the maximum approximately 15% greater than the 90th percentile. We set the mode to \( m = C_j^{\text{nom}} \). Finally, to be able to compare with the Uniform distributions, we set the variance of the Beta1 distribution equal to that of Unif1 and that of Beta2 to that of Unif2. This allows us to examine the effects of skewness and variance separately. We generate the Beta distributions following Kuhl et al. (2010, pp. 96-97) and obtain the following formulas:
Beta1: $C_j \sim 0.5C_{j}^{\text{nom}} + (1.15C_{j}^{\text{nom}} - 0.5C_{j}^{\text{nom}}) \text{BETA}(17.174, 5.852), \quad (31)$

Beta2: $C_j \sim 0.5C_{j}^{\text{nom}} + (1.15C_{j}^{\text{nom}} - 0.5C_{j}^{\text{nom}}) \text{BETA}(4.011, 1.903), \quad (32)$

where the expression $a + (b - a)\text{BETA}(\alpha_1, \alpha_2)$ denotes a Beta random variable on the interval $[a, b]$ with first shape parameter $\alpha_1$ and second shape parameter $\alpha_2$ as in Kuhl et al. (2010, pp. 82-87). We then use each of these four distributions to generate 1000 independent realizations of capacity values for each county for equity deviation limits $K \in \{0, 0.002, 0.004\}$. For each instance generated, we solve the model and record the bottleneck county.

5.2.2. Results of the probabilistic sensitivity analysis

We find that Wilson (Wi), Granville (Gr) and Wayne (Wa) counties have the highest probabilities of becoming bottlenecks. In Section 5.1, we showed that these counties had the smallest CD ratios for the nominal case as well. Table 2 shows the percentages of instances with those counties as bottlenecks. Counties that were bottlenecks in less than 10% of the instances are included in the “Others (O)” row. Table 2 also shows the summary statistics (mean, standard deviation, minimum and maximum) of the optimal values ($P^*$) of the 1000 instances for $K \in \{0, 0.002, 0.004\}$ for Unif1, Unif2, Beta1 and Beta2 distributions.

For each $K$ value, Wilson County always had the highest bottleneck probability. For lower $K$ values, Granville county has the second highest bottleneck probability under all distributions whereas for higher $K$, Wayne county has the second highest bottleneck probability. The remaining counties have very low bottleneck probabilities in all cases and hence are not shown individually. This analysis is useful to FBCENC by highlighting counties with the highest potential to be bottlenecks and hence the greatest need for capacity expansion. It shows that in the presence of counties with similar MCD ratios, concentrating exclusively on the county with the lowest nominal MCD ratio is not sufficient; capacity improvements need to be considered for all counties with substantial probability of becoming a bottleneck, in this case Wilson, Granville and Wayne.

Table 2. Probabilities of becoming bottleneck county and summary statistics for $P^*$ values (1000 * pounds of food) for different capacity distributions and $K$ limits.

<table>
<thead>
<tr>
<th>$K$=0</th>
<th>$K$=0.002</th>
<th>$K$=0.004</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$=0</td>
<td>$K$=0.002</td>
<td>$K$=0.004</td>
</tr>
<tr>
<td>Unif1</td>
<td>Unif2</td>
<td>Beta1</td>
</tr>
<tr>
<td>Wi</td>
<td>50.9%</td>
<td>40.9%</td>
</tr>
<tr>
<td>Gr</td>
<td>41.4%</td>
<td>40.6%</td>
</tr>
<tr>
<td>Wa</td>
<td>7.7%</td>
<td>15.3%</td>
</tr>
</tbody>
</table>
For all distributions, as $K$ is increased, average $P^*$ values decrease due to increasing MCD ratios. We also observe that the standard deviations of the $P^*$ values are high; suggesting that variability in capacity has a large effect on system effectiveness. Since the $P^*$ values are driven by the minimum of a number of random variables (MCD ratios), the variability of $P^*$ is higher than the variability in the individual MCD ratios. This is due to the bottleneck structure of the problem; one county with a lower MCD ratio than others will reduce the amount that can be shipped to other counties, even if they have substantial additional capacity. Hence accurate capacity estimation for the counties which are at risk of becoming bottlenecks, and focused efforts to increase the lowest MCD ratios among counties are crucial.

The equity measure we use in our study, as expressed in (2), aims to maintain an overall equity level by improving the condition of the county which receives the worst service in terms of equity. Although this is the measure used by FBCENC, other food banks may evaluate their equity levels differently. For this reason, we also examined how the solutions proposed by our Food Distribution Model perform in terms of four alternative measures of inequity discussed by Marsh and Schilling (1994). We let $E_j = \frac{x_j}{\sum_{i=1}^{n} x_i} - \frac{d_j}{\sum_{i=1}^{n} d_i}$ and $\bar{E} = \frac{\sum_{i=1}^{n} E_i}{n}$. As described in Section 3, equations (2.a) and (2.b) imply that the inequity measure used in the Food Distribution Model is equivalent to constraining $\max_j E_j$, the first inequity measure discussed by Marsh and Schilling (1994), to be below a certain limit, $K$. We compared the results from the 1000 instances described above and calculated the resulting values for the following alternative inequity measures: 1) Variance, $\frac{\sum_{j=1}^{n} (E_j - \bar{E})^2}{n}$; 2) Average absolute deviation from $\bar{E}$, $\frac{\sum_{j=1}^{n} |E_j - \bar{E}|}{n}$; 3) The range, $\max_j E_j - \min_j E_j$; and 4) Maximum absolute deviation from $\bar{E}$, $\max_j |E_j - \bar{E}|$. We have scaled the first and second measures so that they all take values between zero and one, where zero corresponds to perfect equity and one to the worst level of equity. Our solutions are optimal for each inequity measure considered for $K = 0$ since it requires that $E_j = 0$ for all $j$. For any $K > 0$, the realizations of the inequity measures were always below $K$ indicating that our solutions behave well under different commonly used equity measures. Appendix D contains details of this analysis.
6. Conclusions and limitations

We study the food distribution network of a food bank that is an affiliate of Feeding America and develop models to determine the equitable and effective distribution of food donations for such a food bank. Our objective is to maximize the amount of distributed food while limiting the absolute deviation from a perfectly equitable distribution for each county in the food bank’s service area. We formulate a linear programming model that maximizes the food bank’s effectiveness while satisfying a user-specified constraint on the magnitude of each county’s deviation from perfect equity. The structure of the optimal solution depends on the MCD ratios and the total supply, allowing identification of bottleneck counties whose capacity must be increased in order to increase effectiveness. This insight leads us to formulate a second model showing how a food bank should allocate extra capacity to the counties in its service area. We prove that a stepwise capacity allocation algorithm gives the optimal allocation of additional capacity, and allows estimation of the amount of additional food that can be shipped for a given amount of additional capacity. We then use the data obtained from FBCENC to present numerical results for the given models.

The policies and numerical results from this study have been shared with FBCENC’s staff. Our results in terms of the bottleneck counties aligned with what they have been observing in their operations and hence, our capacity allocation scheme gave them a useful perspective on how they can allocate their budgets, develop proposals for increasing capacity as well as generate strategies for recruiting new agencies. Our results provide theoretically optimal solutions and have the potential to considerably improve the level of equity while minimizing waste.

The operations and objectives of FBCENC are representative of many food banks in the United States that are affiliates of Feeding America. The problem of the equitable and effective distribution of food under capacity limitations is faced by many food banks that collect donations from multiple sources and distribute food to their partner agencies. Furthermore, Feeding America faces a similar problem on a larger scale, i.e. the equitable and effective distribution of food among the food banks that have limited capacities throughout the entire nation.

An important assumption in this paper is that demand and capacity are deterministic. This assumption is appropriate here as our purpose is to understand the general behavior and structural properties of this network-flow problem, especially the interactions between the objectives of equity and effectiveness and the roles of capacity and demand. Our sensitivity analysis shows that capacity variation at the bottleneck counties has an effect on the optimal amount of unshipped food due to the bottleneck nature of the problem. This highlights the importance of accurate capacity estimation and capacity enhancement at the bottleneck
counties. Our future work will relax these assumptions to incorporate the stochastic nature of the models to understand how policies more robust to capacity variation can be obtained.

The assumption of no-cost interbranch flows is another direction in which our models can be extended. FBCENC’s current operations support this assumption since most donations are collected at the Raleigh branch and then shipped to the other branches for distribution to the counties according to the fair-share criterion. An interesting question for future work is the following: how does the modeling and solution structure presented in this paper change if donations are collected at different branches, where each branch serves a set of counties and flows between those branches are incurred at a cost? Our preliminary analysis shows that in such a situation, the structure of the linear programming model and the optimal solutions remain similar; however, in addition to the capacity or supply constrained solutions identified in Proposition 1, solutions may also be constrained by the ratio of a particular branch’s supply to the total demand it serves, depending on the relative levels of transshipment and waste costs. Moreover in the case that $K > 0$, among the multiple optimal solutions, we would prefer to use the solutions that minimize the interbranch flow costs.

In this paper we have limited our analysis to a single food type; the interactions and capacity substitutions between different food groups will be considered in future work. The extension of the models developed here to multiple time periods is also an interesting direction for future research, as is the inclusion of budget constraints on operating costs.

**Acknowledgements**

This research has been supported by the National Science Foundation with the grant numbers #CMMI-1000018 and #CMMI-1000828. The opinions expressed in the paper represent those of the authors and not necessarily those of the National Science Foundation. We would also like to thank the Food Bank of Central and Eastern North Carolina; specifically, Charlie Hale, Vice President of IT & Operations at FBCENC and Earline E. Middleton, Vice President of Agency Services and Programs at FBCENC for their help and support.

**References**


APPENDIX A

Proposition 1. An optimal solution to a given instance of the Food Distribution Model is as follows:

Case 1: If $\sum_{l=1}^{n} C_l \geq S$ and $J_0 = \emptyset$, i.e., the instance is partially equity constrained, then the optimal objective function value is:

$$ P^* = 0, \quad (31) $$

and the individual distributions to the counties $(X_j^*)$ have multiple optimal solutions.

Case 2: If $J_0 \neq \emptyset$ and $\sum_{l=1}^{n} C_l \geq S \geq R \sum_{l=1}^{n} D_l$, i.e., the instance is capacity and equity constrained, then the optimal objective function value is:

$$ P^* = S - R \sum_{l=1}^{n} D_l. \quad (32) $$

In an optimal solution, the bottleneck counties receive an amount of food equal to their capacity,

$$ X_j^* = C_j \text{ for all } j \in B. \quad (33) $$

Food shipments to all remaining counties will have multiple optimal solutions.

Case 3: If $J_0 \neq \emptyset$, $\sum_{l=1}^{n} C_l \geq S$, and $R \sum_{l=1}^{n} D_l > S$, i.e., the instance is supply and equity constrained, then the optimal objective function value is:

$$ P^* = 0. \quad (34) $$

The individual distributions to the counties $(X_j^*)$ will have multiple optimal solutions.

Proof of Proposition 1. The proof will consider each of the three cases separately. Let $\Delta = \sum_{l=1}^{n} D_l$. 

Proof
Case 1: In this case we have $\frac{D_j}{\Delta} \leq K$ for all counties $j \in J$. This implies that $-K + \frac{D_j}{\Delta} \leq 0$ for all $j$, i.e., $J_0 = \emptyset$. Constraint (2) can be written as:

$$-K + \frac{D_j}{\Delta} \leq \frac{X_j}{\sum_{l=1}^{n} X_l} \leq K + \frac{D_j}{\Delta} \quad j \in J. \quad (A1)$$

which, due to the nonnegativity of $X_j$, (A1) can be rewritten as:

$$0 \leq \frac{X_j}{\sum_{l=1}^{n} X_l} \leq K + \frac{D_j}{\Delta} \quad j \in J. \quad (A2)$$

Combining (A2) with the capacity constraint (4) yields

$$0 \leq X_j \leq \min \left( C_j, \left( K + \frac{D_j}{\Delta} \right) \sum_{l=1}^{n} X_l \right) \quad j \in J, \quad (A3)$$

where constraints (3) and (5) ensure that

$$\sum_{l=1}^{n} X_l \leq S. \quad (A4)$$

We will prove that there always exists a feasible solution set $\{X^*_j, j \in J\}$ such that $\sum_{l=1}^{n} X^*_l = S$.

For the given value of $S$, let

$$X_j = \min \left( C_j, S \frac{D_j}{\Delta} \right) \quad j \in J. \quad (A5)$$

Let

$$X_C \equiv \{ j \in J: X_j = C_j \}, \quad (A6)$$

$$X_S \equiv \{ j \in J: X_j = S \frac{D_j}{\Delta} \}. \quad (A7)$$

If, $X_C = \emptyset$ and $X_S \neq \emptyset$, i.e., for all $j$, $X_j = S \frac{D_j}{\Delta}$; then $\sum_{l=1}^{n} X_l = S$ and an optimal solution has been obtained.
Next, we show that the case when $X_S = \emptyset$ and $X_C \neq \emptyset$ is not possible. This case would imply that for all $j$, $C_j < S \frac{D_j}{\Delta}$. Summing over all $j$, we get $\sum_{i=1}^n C_i < S$, which is a contradiction since we assume that $S \leq \sum_{i=1}^n C_i$.

The last case we consider is $X_S \neq \emptyset$ and $X_C \neq \emptyset$. For counties $l \in X_C$, define

$$E_l = S \frac{D_l}{\Delta} - C_l \quad l \in X_C.$$  \hspace{1cm} (A8)

The value $E_l$ represents the extra amount of supply that must be allocated to some county $j \in X_S$ with idle capacity in order to achieve $\sum_{i=1}^n X_i = S$. The total idle capacity available must be greater than or equal to the total extra pounds of food to be shipped in order for this solution to hold, implying that

$$\sum_{l \in X_C} E_l \leq \sum_{l \in X_S} \left( C_l - S \frac{D_l}{\Delta} \right),$$  \hspace{1cm} (A9)

where $C_l - S \frac{D_l}{\Delta}$ represents the idle capacity at county $l \in X_S$.

Since we have $S \leq \sum_{i=1}^n C_i$ by assumption,

$$S \left( \sum_{l \in X_C} \frac{D_l}{\Delta} + \sum_{l \in X_S} \frac{D_l}{\Delta} \right) \leq \sum_{l \in X_C} C_l + \sum_{l \in X_S} C_l,$$  \hspace{1cm} (A10)

$$S \sum_{l \in X_C} \frac{D_l}{\Delta} - \sum_{l \in X_S} C_l \leq \sum_{l \in X_S} C_l - S \sum_{l \in X_S} \frac{D_l}{\Delta},$$  \hspace{1cm} (A11)

$$\sum_{l \in X_C} E_l \leq \sum_{l \in X_S} C_l - S \sum_{l \in X_S} \frac{D_l}{\Delta},$$  \hspace{1cm} (A12)

so inequality (A9) always holds.

Therefore, by assigning this extra $\sum_{l \in X_C} E_l$ pounds of food among the counties in set $X_S$ in an arbitrary manner, we can obtain an optimal solution with $\sum_{i=1}^n X_i^* = S$. Furthermore, since there
can be different assignments to the counties with idle capacity, there can be multiple optimal solutions to the Food Distribution Model for Case 1. An algorithm for generating these alternative allocations is given in Figure 3 of the paper.

**Case 2:** For Case 2 we have at least one county $j \in J$ such that $\frac{D_j}{\Delta} > K$, and $S \geq R\Delta$. Hence, constraint (2), in combination with the capacity constraint (4), can be written as:

$$\max\left(0, \left(-K + \frac{D_j}{\Delta}\right) \sum_{l=1}^{n} X_l\right) \leq X_j \leq \min\left(C_j, \left(K + \frac{D_j}{\Delta}\right) \sum_{l=1}^{n} X_l\right) \quad j \in J_0. \quad (A13)$$

where constraints (3) and (5) ensure that

$$\sum_{l=1}^{n} X_l \leq S. \quad (A14)$$

For feasibility, we must have

$$\left(-K + \frac{D_j}{\Delta}\right) \sum_{l=1}^{n} X_l \leq C_j \quad j \in J_0, \quad (A15)$$

$$\sum_{l=1}^{n} X_l \leq \frac{C_j \Delta}{D_j - K\Delta} \quad j \in J_0, \quad (A16)$$

$$\sum_{l=1}^{n} X_l \leq \min_{j \in J_0} \left\{ \frac{C_j \Delta}{D_j - K\Delta} \right\} \quad (A17)$$

$$\sum_{l=1}^{n} X_l \leq R\Delta. \quad (A18)$$

Since this case satisfies the condition that $S \geq R\Delta$, we only need $\sum_{l=1}^{n} X_l \leq R\Delta$.

We will prove that there always exists a feasible solution set $\{X^*_j, j \in J\}$ that satisfies $\sum_{l=1}^{n} X^*_l = R\Delta$.

For the given value of $R\Delta$, let
\[ X_j = \min(C_j, RD_j) \quad j \in J. \quad \text{(A19)} \]

Let
\[ X_C \equiv \{ j \in J : X_j = C_j \} \quad \text{(A20)} \]
\[ X_S \equiv \{ j \in J : X_j = RD_j \}. \quad \text{(A21)} \]

If, \( X_C = \emptyset \) and \( X_S \neq \emptyset \), i.e., for all \( j \), \( X_j = RD_j \), then \( \sum_{l=1}^{n} X_l = R\Delta \) is obviously an optimal and feasible solution.

Next, we show that the case when \( X_S = \emptyset \) and \( X_C \neq \emptyset \) is not possible. This case would imply that for all \( j \), \( C_j < RD_j \). Summing over all \( j \), we get \( \sum_{l=1}^{n} C_l < R\Delta \leq S \) due to the assumption of Case 2. This is a contradiction since we assume that \( S \leq \sum_{l=1}^{n} C_l \).

The last case we consider is \( X_S \neq \emptyset \) and \( X_C \neq \emptyset \). For counties \( l \in X_C \), define
\[ E_l = RD_l - C_l \quad l \in X_C. \quad \text{(A22)} \]
The value \( E_l \) represents the extra amount of supply to be allocated to any county \( j \in X_S \) with idle capacity in order to achieve \( \sum_{l=1}^{n} X_l = R\Delta \). This \( E_l \) pounds of food can be allocated. The total idle capacity available should be greater than or equal to the total extra pounds of food to be shipped in order for this solution to hold. So, we must have:
\[ \sum_{l \in X_C} E_l \leq \sum_{l \in X_S} (C_l - RD_l), \quad \text{(A23)} \]
where \( C_l - RD_l \) represents the idle capacity at county \( l \in X_S \).

By using the main assumption of \( S \leq \sum_{l=1}^{n} C_l \) and the condition of this case that \( S \geq R\Delta \), we get \( \sum_{l=1}^{n} C_l \geq R\Delta \). It follows that,
\[ R \left( \sum_{l \in X_C} D_l + \sum_{l \in X_S} D_l \right) \leq \sum_{l \in X_C} C_l + \sum_{l \in X_S} C_l, \quad \text{(A24)} \]
\[ R \sum_{t \in X_c} D_t - \sum_{t \in X_c} C_t \leq \sum_{t \in X_s} C_t - R \sum_{t \in X_s} D_t, \quad (A25) \]

\[ \sum_{t \in X_c} E_t \leq \sum_{t \in X_s} C_t - R \sum_{t \in X_s} D_t. \quad (A26) \]

So, inequality (A23) always holds.

Therefore, by assigning this extra \( \sum_{t \in X_c} E_t \) pounds of food to the counties in set \( X_s \) in an arbitrary manner, we can obtain the optimal solution of \( \sum_{i=1}^{n} X_i^* = R \Delta \). Furthermore, since there can be different assignments to the counties with idle capacity, this shows that there can be multiple optimal solutions to the Food Distribution Model for Case 2.

Proof for distribution to bottleneck counties: According to the definition of a bottleneck county given in Proposition 1, bottleneck counties are those with the minimum \( \frac{C_j}{D_j-K\Delta} \) ratio among the counties \( j \in J_0 \). Then, for \( j \in B \), since we have shown that \( \sum_{i=1}^{n} X_i^* = R \Delta \), from Constraint (2), we have

\[
\left( -K + \frac{D_j}{\Delta} \right) R \Delta \leq X_j^* \leq \min \left( C_j, \left( K + \frac{D_j}{\Delta} \right) R \Delta \right) \quad j \in B, \quad (A27)
\]

\[
\left( -K + \frac{D_j}{\Delta} \right) \left( \frac{C_j \Delta}{D_j - K \Delta} \right) \leq X_j^* \leq \min \left( C_j, \left( K + \frac{D_j}{\Delta} \right) \left( \frac{C_j \Delta}{D_j - K \Delta} \right) \right) \quad j \in B, \quad (A28)
\]

\[ C_j \leq X_j^* \leq C_j \quad j \in B. \quad (A29) \]

It follows that

\[ X_j^* = C_j \text{ for } j \in B. \quad (A30) \]

Case 3: Case 3 considers the situation where there exists at least one county \( j \in J \) such that \( \frac{D_j}{\Delta} > K \), and \( S < R \Delta \). The proof for this case follows from a combination of the proofs for Cases 1 and 2. We can apply the equations (A13)-(A18) exactly to this case. However, due to the condition of
that this case satisfies, we only need $\sum_{i=1}^{n} X_i \leq S$. The remaining argument follows along the lines of Case 1 as given in (A5)-(A12).

APPENDIX B

Proof of Proposition 2. Let $\Delta = \sum_{i=1}^{n} D_i$. If we associate dual variables $\pi_i$ with the constraints (18) and $\pi_0$ with constraint (19), then the dual of the Capacity Allocation Model can be written as

$$\min \sum_{i \in J_0} C_i \pi_i + \pi_0$$

(B1)

$$\sum_{i \in J_0} (D_i - K\Delta) \pi_i \geq 1$$

(B2)

$$-Y \pi_j + \pi_0 \geq 0 \quad j \in J_0$$

(B3)

$$\pi_j \geq 0 \quad j \in J_0 \cup \{0\}$$

(B4)

From (B3), we see that

$$\pi_j \leq \frac{\pi_0}{Y} \quad j \in J_0$$

(B5)

and thus from (B2), we have

$$\sum_{i \in J_0} \frac{(D_i - K\Delta) \pi_0}{Y} \geq \sum_{i \in J_0} (D_i - K\Delta) \pi_j \geq 1.$$  

(B6)

From (B6), we have
\[ \pi_0 \geq \frac{Y}{\sum_{l \in J_0} (D_l - K\Delta)} > 0 \] (B7)

and therefore by the complementary slackness theorem (Bertsimas & Tsitsiklis, 1997) constraint (19) must be satisfied at strict equality. ■

APPENDIX C

Proof of Proposition 3. Let \( \Delta = \sum_{l=1}^{n} D_l \). Direct inspection of the Capacity Allocation Algorithm suggests that the solution obtained from the Capacity Allocation Algorithm is optimal. Here, we will use the solution obtained from the algorithm and prove that it is optimal. By the operation of the Capacity Allocation Algorithm, if we terminate the algorithm with \( \eta \) bottleneck counties, the algorithm must terminate at iteration \( \eta \). All bottleneck counties have MCD ratios of at least \( \frac{C_l}{D_{\eta} - K\Delta} \).

Since the algorithm terminated at iteration \( \eta \), there was not sufficient capacity to perform the next iteration, so the optimal number of bottleneck counties is

\[
\eta = \arg\max_{1 \leq \xi \leq |J_0|} \left\{ \sum_{l=1}^{\xi-1} \left( \frac{C_l (D_l - K\Delta)}{D_{\xi} - K\Delta} - C_l \right) \leq Y \right\} \] (25)
In order to show how equation (25) is obtained, let \( W_j \) denote the total additional capacity to be allocated to county \( j \) as a result of the Capacity Allocation Algorithm where \( j \leq \eta \). Then, to reach iteration \( \eta > 1 \), we need

\[
\frac{C_j + W_j}{D_j - K\Delta} = \frac{C_\eta}{D_\eta - K\Delta} \quad j < \eta
\] (C1)

\[
W_j = \frac{C_\eta(D_j - K\Delta)}{D_\eta - K\Delta} - C_j \quad j < \eta
\] (C2)

This implies that the total additional capacity needed to reach iteration \( \eta \) is:

\[
\sum_{\ell=1}^{\eta-1} W_\ell = \sum_{\ell=1}^{\eta-1} \left( \frac{C_\eta(D_\ell - K\Delta)}{D_\eta - K\Delta} - C_\ell \right)
\] (C3)

and should satisfy

\[
\sum_{\ell=1}^{\eta-1} W_\ell = \sum_{\ell=1}^{\eta-1} \left( \frac{C_\eta(D_\ell - K\Delta)}{D_\eta - K\Delta} - C_\ell \right) \leq Y.
\] (C4)

The equation (25) follows directly. The summation in (25) is taken to be zero when \( \xi = 1 \), from which it follows that we stop at iteration \( \eta = 1 \).

The solution \([Q^*, (\rho_j^*; j \in J_0)]\), as given in Proposition 3 is feasible for the Capacity Allocation Model. Let \([(\pi_j, j \in J_0), \pi_0]\) represent the corresponding dual solution. The vectors \([Q^*, (\rho_j^*; j \in J_0)]\) and \([(\pi_j, j \in J_0), \pi_0]\) are optimal solutions for the two respective problems if and only if, by the Complementary Slackness Theorem (Bertsimas & Tsitsiklis, 1997), \([(\pi_j, j \in J_0), \pi_0]\) is a feasible dual solution and they satisfy the following:

\[
\pi_j(Q^*D_j - KQ\Delta - \rho_j^*Y - C_j) = 0 \quad j \in J_0
\] (C5)

\[
\pi_0 \left( \sum_{i \in J_0} \rho_i^* - 1 \right) = 0
\] (C6)
\[
\left( \sum_{l \in J_0} (D_l - K\Delta)\pi_l - 1 \right) Q = 0 \tag{C7}
\]

\[
(-Y\pi_j + \pi_0)\rho_j^* = 0 \quad j \in J_0 \tag{C8}
\]

Since by Proposition 2, Constraint (19) is satisfied at equality in an optimal solution, we obtain no additional information from (C6).

Assume that the number of bottleneck counties obtained from (25) is \( \eta \). From (C5), using the proposed optimal solution in Proposition 3, for \( j \leq \eta \),

\[
Q^* D_j - KQ^*\Delta - \rho_j^* Y - C_j = Q^* D_j - KQ^*\Delta - \frac{Q^*(D_j - K\Delta) - C_j}{Y} Y - C_j = 0. \tag{C9}
\]

From Equations (B4) and (C5),

\[
\pi_j \geq 0 \text{ for } 1 \leq j \leq \eta. \tag{C10}
\]

For \( \eta + 1 \leq j \leq n \), from Equation (C5) and using the proposed optimal solution in Proposition 3,

\[
Q^* D_j - KQ^*\Delta - \rho_j^* Y - C_j = \frac{D_j (Y + \sum_{g=1}^{\eta} C_g) - K\Delta (Y + \sum_{g=1}^{\eta} C_g) - C_j}{\sum_{l=1}^{\eta} (D_l - K\Delta)} \sum_{l=1}^{\eta} (D_l - K\Delta) \tag{C11}
\]

Based on the termination condition of the Capacity Allocation Algorithm, if \( \eta < |J_0| \), then we have

\[
\frac{C_{\eta+1}}{D_{\eta+1} - K\Delta} > \ldots > \frac{C_{\eta+1}}{D_{\eta+1} - K\Delta} \sum_{l=1}^{\eta} (D_l - K\Delta) = R_\eta; \tag{C12}
\]

and therefore we have,

\[
\frac{C_j}{D_j - K\Delta} > \frac{Y + \sum_{l=1}^{\eta} C_l}{\sum_{l=1}^{\eta} (D_l - K\Delta)} \text{ for } \eta + 1 \leq j \leq |J_0|. \tag{C13}
\]

It follows that
\[
D_j \left( Y + \sum_{l=1}^{\eta} C_l \right) - K\Delta \left( Y + \sum_{l=1}^{\eta} C_l \right) - C_j \sum_{l=1}^{\eta} (D_l - K\Delta) < 0 \text{ for } \eta + 1 \leq j
\]
\[
\leq |J_0|.
\]

Since this is the numerator of equation (C11), from (C5), it follows that
\[
\pi_j = 0 \text{ for } \eta + 1 \leq j \leq |J_0|.
\]

(C14)

If \( \eta = |J_0| \), equations (C11) – (C15) are not needed.

We can assume that \( Q > 0 \) because it represents the minimum MCF ratio after capacity allocation.

Then, from (C7), we have
\[
\sum_{l \in J_0} (D_l - K\Delta)\pi_l = 1.
\]

(C16)

Using (C15), we can rewrite (C16) as
\[
\sum_{l=1}^{\eta} (D_l - K\Delta)\pi_l = 1.
\]

(C17)

By using the proposed solution, without loss of generality, we can assume that \( \rho_j \geq \varepsilon \) for \( j \leq \eta \), where \( \varepsilon \) is a small positive number. Then, from (C6), it follows that
\[
-Y\pi_j + \pi_0 = 0 \text{ for } j \leq \eta
\]
\[
\pi_j = \frac{\pi_0}{Y} \text{ for } j \leq \eta
\]

(C18)

(C19)

By inserting (C19) into (C17),
\[
\sum_{l=1}^{\eta} (D_l - K\Delta)\frac{\pi_0}{Y} = \frac{\pi_0}{Y} \sum_{l=1}^{\eta} (D_l - K\Delta) = 1
\]

(C20)
\[
\pi_0 = \frac{Y}{\sum_{l=1}^{\eta} (D_l - K\Delta)}.
\] (C21)

By inserting (C21) into (C19), we get

\[
\pi_j = \frac{1}{\sum_{l=1}^{\eta} (D_l - K\Delta)} \quad \text{for } j \leq \eta.
\] (C22)

This result is also intuitively meaningful; \(\pi_j\) is the marginal benefit of increasing \(C_j\) by one unit. If we examine the structure of \(Q^*\) in Equation (26), we can see that if \(C_j\) for \(j \leq \eta\) is increased by one unit, \(Q^*\), which is the optimal objective function value, increases by \(\frac{1}{\sum_{l=1}^{\eta} (D_l - K\Delta)}\). This solution, \([\pi_j, j \in J_0], \pi_0\], as given by equations (C14), (C20) and (C21) is also feasible for the dual problem since it satisfies constraints (B2)-(B4).

Finally, calculating the dual objective function value,

\[
Z_{dual} = \sum_{l \in J_0} C_l \pi_l + \pi_0 = \frac{\sum_{l=1}^{\eta} C_l}{\sum_{l=1}^{\eta} (D_l - K\Delta)} + \frac{Y}{\sum_{l=1}^{\eta} (D_l - K\Delta)} = Q^* = Z_{primal}
\] (C22)

Hence, by the Strong Duality Theorem (Bertsimas and Tsitsiklis, 1997), the proposed solution is optimal. ■

APPENDIX D
Analysis of different equity measures: We examine how the solutions proposed by our Food Distribution Model perform in terms of four alternative measures of inequity discussed by Marsh and Schilling (1994). First, we make the following definitions in accordance with Marsh and Schilling (1994):

\[
E_j = \left| \frac{X_j}{\sum_{l=1}^{n} X_l} - \frac{D_j}{\sum_{l=1}^{n} D_l} \right|
\]

(D1)

\[
\bar{E} = \frac{\sum_{l=1}^{n} E_l}{n}
\]

(D2)

As described in Section 3, through equations 2.a and 2.b, the inequity measure used in the Food Distribution Model is equivalent to constraining \(\max_j E_j\), which is the first inequity measure discussed by Marsh and Schilling (1994), to be below a certain limit. By doing this, we enforce that this inequity measure remains below an equity deviation limit, \(K\). We will compare our results from the Food Distribution Model to four alternative inequity measures. The measures we will use are: 1) Variance, \(\frac{\sum_{j=1}^{n}(E_j - \bar{E})^2}{n}\); 2) Average absolute deviation from \(\bar{E}\), \(\frac{\sum_{j=1}^{n}|E_j - \bar{E}|}{n}\); 3) The range, \(\max_j E_j - \min_j E_j\); and 4) Maximum absolute deviation from \(\bar{E}\), \(\max_j |E_j - \bar{E}|\). We have scaled some of the measures from Marsh and Schilling (1994) to normalize the inequity measures so that they all take values between zero and one. Since these are all measures of inequity, smaller values indicate a better equity level.

In terms of the experimental design, we use the same approach as explained in the previous section for uncertainty in capacities. We will again use the equity deviation limits, \(K = 0, 0.002, 0.004\) since we would like to select \(K\) values corresponding to capacity and equity constrained instances. We then use the obtained optimal solutions for each instance and calculate the inequity levels for each of the four measures considered. The average inequity levels from the
1000 instances for Beta2 distribution are summarized in Table D1 where the values in parentheses show the corresponding standard deviations. The remaining distributions are not shown here since this distribution has the highest level of variance and skewness among the considered distributions and hence exhibits the widest variability.

**Table D1.** Analysis of different inequity measures from Marsh and Schilling (1994) for Beta2 distribution.

<table>
<thead>
<tr>
<th>Inequity Measures</th>
<th>0</th>
<th>0.002</th>
<th>0.004</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\sum_{j=1}^{n}(E_j - \bar{E})^2}{n} )</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>( \frac{\sum_{j=1}^{n}</td>
<td>E_j - \bar{E}</td>
<td>}{n} )</td>
<td>0.00000</td>
</tr>
<tr>
<td>( \max_j E_j - \min_j E_j )</td>
<td>0.00000</td>
<td>0.00073</td>
<td>0.00242</td>
</tr>
<tr>
<td>( \max_j</td>
<td>E_j - \bar{E}</td>
<td>)</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

The inequity measure used in the Food Distribution Model limits all \( E_j \) values to stay below a certain limit and hence, forces a certain equity level on each county. The perfect equity case, \( K = 0 \) requires that \( E_j = 0 \) for all \( j \). Hence, all the other measures are also equal to zero indicating that our solutions are optimal for each inequity measure considered for \( K = 0 \). When \( K > 0 \), each \( E_j \) is required to be below \( K \), and hence \( \bar{E} \) is also required to be less than \( K \). This causes all the measures containing the \( (E_j - \bar{E}) \) term to achieve low levels. The measure that behaves the worst is the range, \( \max_j E_j - \min_j E_j \), but that measure is also constrained to be lower than the original measure since \( \min_j E_j > 0 \). The results show that the measure we use in our paper is a very strong...
equity measure and enforces a certain level of equity at each of the locations considered. This causes the resulting policies to behave well under different commonly used equity measures.

References
